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When are synchronization errors small?

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We address the question of bounds on the synchronization error for the case of nearly identical nonlinear systems. It is pointed out that negative largest conditional Lyapunov exponents of the synchronization manifold are not sufficient to guarantee a small synchronization error and that one has to find bounds for the deformation of the manifold due to perturbations. We present an analytic bound for a simple subclass of systems, which includes the Lur'e systems, showing that the bound for the deformation grows as the largest singular value of the linearized system gets larger. Then, the Lorenz system is taken as an example to demonstrate that the phenomenon is not restricted to Lur'e systems.

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I. INTRODUCTION

Chaos synchronization [1,2] has attracted much attention during the past years because of its role in our understanding of the basic features of man-made and natural systems. For example, optical communication with chaotic wave forms, demonstrated both experimentally [3] and theoretically [4], are only possible because of chaos synchronization between receiver and transmitter.

In this paper we consider the synchronization of two nearly identical nonlinear systems that are unidirectionally coupled. Let us denote the driver or transmitter by x and the response system or receiver by $\tilde{x}$. In the control literature the problem of synchronization under unidirectional coupling is connected to that of the observability of x. If the response system $\tilde{x}$ synchronizes to x, then we will have full knowledge about the state of the drive system; it becomes observable and the response system is called the observer.

For unidirectional coupling between identical systems, one is usually trying to achieve identical synchronization, meaning $x(t)=\tilde{x}(t)$, which is what we consider. If we assume that the drive and the response systems are identically synchronized when their parameters are identical, then under slight parameter mismatch they will loose identical synchronization but might still exhibit generalized synchronization, meaning that there exists a functional relationship between states of the drive and response. Sufficient conditions for the stability and smoothness of the identical synchronization manifold can be given in terms of Lyapunov exponents [5–9].

In practice, one is, however, often interested in the deviation from identical synchronization if small parameter mismatches are present or other small perturbations such as noise prevent the system from reaching identical synchrony. The reason is that, even if generalized synchronization exists, we usually do not know the relationship between x and $\tilde{x}$, and for that reason cannot determine the state of the driver by observing the state of the response system. One, therefore, has to address the question of how large the synchronization error gets for a given size of the parameter mismatch and what properties of the dynamical systems can be used to obtain a bound on the size of the error, where we define the synchronization error e as the deviation from the identical synchronization manifold, $e(t)=\tilde{x}(t)-x(t)$.

In this paper we address the question of bounds on the synchronization error for the case of nearly identical nonlinear systems. It is pointed out that the negative largest conditional Lyapunov exponents of the synchronization manifold are not sufficient to guarantee a small synchronization error and that one has to find bounds for the deformation of the manifold due to perturbations. The similar problem of how to guarantee high quality synchronization in coupled oscillators has been recently investigated by Blakely *et al.* [10]. This is the outline of the paper. In Sec. II of this paper we derive a bound for the synchronization error for systems where the error dynamics can be written in terms of a driven linear ordinary differential equation. In Sec. III we apply the result of the Sec. II to the case of Lur'e systems and present an example. In Sec. IV we show on the example of two coupled Lorenz systems that the synchronization error grows with the singular values for more general nonlinear systems not covered in Sec. II.

II. ERROR BOUNDS

We consider two nearly identical unidirectionally coupled systems that may be written as

$$\dot{x}=f_p(x), \qquad (1)$$

$$\dot{\tilde{x}}=f_{\tilde{p}}(\tilde{x})+C(x,\tilde{x}-x),$$

where $p,\tilde{p}$ are the different parameters, x and $\tilde{x}$ are d-dimensional vectors, and $C(x,\tilde{x}-x)$ is the coupling function with $C(x,0)=0$.

Before we give and discuss an error bound for a specific subclass of systems, (1), let us specify the assumptions we

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make and how the question we address in this paper relates to well-known stability conditions in terms of Lyapunov exponents.

We assume that the driving system $\dot{x}=f_p(x)$ has an attractor $\mathcal{A}$, which may or may not be chaotic. To every ergodic measure supported in $\mathcal{A}$ there correspond d Lyapunov exponents, which are usually called tangential. We write μ_{min} for the smallest tangential Lyapunov exponent, smallest considering all probability measures supported on the attractor $\mathcal{A}$. In the case of identical parameters $p=\tilde{p}$, the synchronization manifold is given by $x=\tilde{x}$. The stability analysis of the synchronization state $s(t)=x=\tilde{x}$ leads to a linear variational equation of the form

$$\dot{e}=[Df_p(s)+DC(s)]e, \tag{2}$$

where Df_p is the Jacobian of the vector field f and DC is the Jacobian of the coupling function C both evaluated on $s(t)$. Equation (2) can be recognized as the variational equation used to compute the Lyapunov exponents of the response, $\lambda_i(\rho)(i=1,\dots,d)$, which are often referred to as normal Lyapunov exponents. We write λ_{max} for the largest normal Lyapunov exponent, again considering all probability measures supported on the attractor as, for instance, those given by the unstable periodic orbits in $\mathcal{A}$. The identical synchronization (IS) manifold $x=\tilde{x}$ is stable iff [6]

$$\lambda_{max}<0. \tag{3}$$

In a real system we will always have $p\neq\tilde{p}$. In this case the synchronization manifold is not $x=\tilde{x}$ but for $p\approx\tilde{p}$ takes the form of a generalized synchronization (GS) manifold of the form $\tilde{x}=F(x)$, under the assumption that the receiver system is asymptotically stable for all initial $x(0)$ in a neighborhood of the chaotic attractor $\mathcal{A}$ of the driver [11]. The smoothness of the perturbed manifold (GS manifold) depends on the normal hyperbolicity of the IS manifold. The condition for normal hyperbolicity can be expressed in terms of Lyapunov exponents [5,7,9], as

$$\lambda_{max}<\mu_{min}. \tag{4}$$

If the contraction toward the synchronization manifold is sufficiently strong and if this is true for all trajectories embedded in the (chaotic) attractor $\mathcal{A}$, then the manifold will be persistent under perturbations. The smoothness of the GS manifold guarantees that global properties like the dimension of the attractor will be preserved. In Ref. [9] we present an example showing that when Eq. (4) is not satisfied, the GS manifold may be a fractal (in addition to the fractal nature of the chaotic attractor of the driver).

Besides the question of smoothness of the synchronization manifold under parameter mismatch one can ask under what condition is the GS manifold close to the IS manifold. This second question is concerned with the deviation of phase space trajectories of the synchronized system (generalized synchronization) from the IS manifold, which is of practical importance because we usually have no access to the functional form of the map $F(x)$ which defines the GS manifold. To address this question we consider the equation for the error dynamics of the form

$$\dot{e}=f_{\tilde{p}}(e+x)-f_p(x)+C(x,e), \tag{5}$$

and rewrite it so that the linear, nonlinear, and driving terms are explicitly separated. To that avail we Taylor expand both f and C as $f_{\tilde{p}}(e+x)=f_{\tilde{p}}(x)+Df_{\tilde{p}}(x)e+g_f(x,e)$ and $C(x,e)=D_2C(x,0)e+g_C(x,e)$, where D_2C denotes the derivative of the coupling function $C(x,e)$ with respect to the second argument. All terms of higher order in e are lumped into the functions g_f and g_C, both that vanish for $e=0$. Equation (5) then reads

$$\begin{aligned}\dot{e}&=[Df_{\tilde{p}}(x)+D_2C(x,0)]e+[f_{\tilde{p}}(x)-f_p(x)]\\&\quad+[g_f(x,e)+g_C(x,e)]\\&\equiv A(t)e+h(t)+g(t,e),\end{aligned} \tag{6}$$

where $A(t)$ is a $d\times d$ matrix with time-varying coefficients, $g(t,e)$ describes nonlinear terms in the error, $g(t,0)=0$, and $h(t)$ is a driving term due to the parameter mismatch. We assume that both Eqs. (3) and (4) hold, and that there exists a GS manifold if $p\neq\tilde{p}$.

The linearized equation is

$$\dot{e}=A(t)e+h(t). \tag{7}$$

For typical cases one expects the bound on the synchronization error e to be proportional to the bound on the driving term $h(t)$ and inversely proportional to the rate of contraction toward the IS manifold, e.g., to λ_{max}. However, finding an explicit bound on e as given by Eq. (7) without making any further assumptions on either $h(t)$ or $A(t)$ is a difficult task, and we do not attempt to do this. We instead restrict ourselves to the specific case where A is a constant matrix. This enables us to derive an error bound and to gain insight into the mechanisms that can lead to large synchronization errors. We then will present an example demonstrating that the same effect can be observed for the general case.

Let us start by considering the driven linear differential equation with a constant matrix $A\in\mathbb{R}^{d\times d}$ and a continuous and bounded driving term $h(t)$, $|h(t)|\leqslant H$, $\forall\ t\in[0,\infty]$,

$$\dot{x}(t)=Ax(t)+h(t), \tag{8}$$

which has the general solution

$$x(t)=e^{At}x_0+\int_0^t e^{A(t-s)}h(s)ds. \tag{9}$$

For any square matrix A there exists a unitary (orthogonal) transformation U that brings A into upper right triangular form (Schur form), which we write as the diagonal D containing all the eigenvalues of A and the strictly upper right triangular matrix N with zeros as diagonal elements,

$$A=U(D+N)U^{\dagger},$$

where $U^\dagger$ denotes the Hermitian of U. If at least one of the eigenvalues of A has a positive real part, the solution $x(t)$ will be unbounded. Let η denotes the maximum real part of all eigenvalues λ_i,

$$\eta \equiv \max_{i=1\ldots d} \mathrm{Re}(\lambda_i). \tag{10}$$

For a stable matrix A, meaning that $\eta<0$ holds, the maximum of $|x(t)|$ will be finite. This is the case we want to consider here and we therefore assume $\eta<0$ for the rest of the paper. Since the solution $x(t)$ will be bounded, the question is how large this bound is compared to the bound H on the driving signal. Using Eq. (9), the following relation holds:

$$|x(t)| \leq |e^{At}||x_0| + \int_0^t |e^{A(t-s)}||h(s)|ds, \tag{11}$$

where $|x|$ denotes Euclidean norm of a vector x and $|A|$ denotes the operator norm for a matrix A, i.e., $|A| := \max_x |Ax|/|x|$. An estimation for the exponential term is given by (see Appendix A)

$$|e^{At}| \leq e^{\eta t} \sum_{k=0}^{d-1} \frac{|N|^k}{k!} t^k, \tag{12}$$

which together with the definition of the incomplete Γ function,

$$P(t,\nu) = \frac{1}{\Gamma(\nu)} \int_0^t e^{-s} s^{\nu-1} ds,$$

yields the bound

$$|x(t)| \leq \sum_{k=0}^{d-1} \left[\left| x_0 \right| \frac{|N|^k}{k!} t^k e^{-|\eta| t} + P(|\eta| t, k+1) \left(\frac{|N|}{|\eta|} \right)^k \frac{H}{|\eta|} \right]. \tag{13}$$

In the asymptotic limit $t\to\infty$, this simplifies to

$$|x(\infty)| \leq \frac{\left(\frac{|N|}{|\eta|} \right)^d - 1}{\left(\frac{|N|}{|\eta|} \right) - 1} \frac{H}{|\eta|}. \tag{14}$$

With no driving and A being stable the solution will, in general, initially grow polynomially before decaying to zero at an exponential rate. For $|N| \gg 1$ this initial peak with a maximum size of the order $|N|^{d-1}$ can be quite substantial. We display a typical example in Fig. 1. With driving the solution will show the initial peak, as in the case of no driving, it will then, however, not decay to zero, but remain finite. As can be seen from the bound (14), this finite amplitude of asymptotically remaining oscillations can be large, if $|N|/|\eta| \gg 1$.

A simple upper bound for the maximum error magnitude between two mismatched systems was recently suggested [12]. It is given by the ratio of the bound on the driving term $|h(t)| \leq H$, $\forall\ t \in [0,\infty]$ and the largest averaged normal

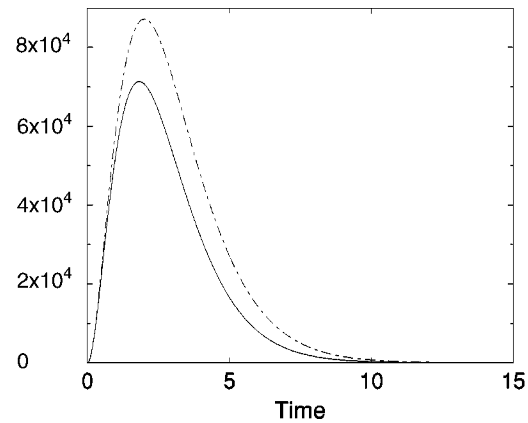

FIG. 1. The solid curve corresponds to the solution of an undriven three-dimensional linear differential equation with a constant matrix A, where $|N|=1000.1$ and $\eta=-1$. We as well display the bound given by Eq. (13) (dashed curve).

Lyapunov exponent as measured at a representative sampling around the attractor. This estimation was suggested for general nonlinear systems where the error dynamics can be written as an equation of form (7). In the case of a constant matrix A, the Lyapunov exponents are just the real part of the eigenvalues λ_i of the matrix A, therefore the suggested bound may in our notation be written as $|x(\infty)| \leq (H/|\eta|)$ (for constant A). This clearly agrees with our result if A is a normal matrix. However, as we can see from the bound we derived Eq. (14) and from the examples presented below for non-normal matrices A there will be a prefactor that is not necessarily negligible, and in this case synchronization errors, large compared to H, can occur. This indicates that, if one is interested in the size of the synchronization error as measured by the Euclidean norm in the given coordinates, one must not only consider the Lyapunov exponents and the size of the parameter mismatch, which determines the bound H, but also the deformation of the identical synchronization manifold. At least in the case of constant matrices A in the error dynamics the effect of the deformation can be estimated by Eq. (14) or in terms of the maximum singular value σ_{max} of A, because for large values of σ_{max}, $|N| \approx \sigma_{max}$ (see Appendix B).

III. LUR'E SYSTEMS

The above analysis applies directly to the synchronization of Lur'e systems.

The class of the so-called Lur'e systems is described by

$$\dot{x} = Ax + \phi(Cx), \tag{15}$$

$$u = Cx,$$

where $x \in \mathbb{R}^d$, $u \in \mathbb{R}$, and A and C are constant matrices of corresponding dimensions. The assumptions are that u is the measured output of the system, the pair (A,C) is observable, and that ϕ is a smooth nonlinear vector field depending only upon the output u. Even if a system is not given in Lur'e form, it is sometimes possible to transform it into the Lur'e form [13]. The Lur'e systems allow the construction of a synchronizing system, also called observer in the control literature, of the form

$$\dot{\tilde{x}} = A\tilde{x} + \phi(Cx) - K(C\tilde{x} - Cx), \tag{16}$$

which, by setting $e = \tilde{x} - x$, yields the error dynamics

$$\dot{e} = (A - KC)e. \tag{17}$$

Clearly, the systems (15) and (16) synchronize if $e=0$ is a stable equilibrium of Eq. (17) or, in other words, if one is able to pick a matrix K so that all eigenvalues of $A-KC$ have negative real parts. This is always possible if A,C satisfy the observability rank condition

$$\text{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{d-1} \end{bmatrix} = d, \tag{18}$$

in which case we say the pair (A,C) is observable.

In general the Lur'e system will depend on some parameters p, which might not be known exactly or which may not be implemented accurately enough in the driven system (16), the observer. One should therefore consider small parameter mismatches. The constant matrix and the nonlinear function of system (16) will differ somewhat from those of the driving system (15), and we will denote them with $\tilde{A}$ and $\tilde{\phi}$, respectively. This leads to a driving term in the error dynamics, and brings it in the form of Eq. (8),

$$\begin{aligned} \dot{e} &= (\tilde{A} - KC)e + (\tilde{A} - A)x + \tilde{\phi}(Cx) - \phi(Cx) \\ &= (\tilde{A} - KC)e + h(x(t)). \end{aligned} \tag{19}$$

The error bound of Sec. II applies therefore to the Lur'e systems. We expect large synchronization errors for small but finite parameter mismatches if the matrix $(\tilde{A}-KC)$ has a large maximum singular value.

Let us demonstrate the main points of what we have said so far with an example. Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & A_{13} \\ -1 & 0 & 0 \\ 0 & A_{32} & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \phi(x_1) \\ -p \\ 0 \end{pmatrix}, \tag{20}$$

with the nonlinear function $\phi(x_1) = 1 - \frac{1}{3}(x_1+1)^3$. We design a response system as described above, so that the error dynamics are given by Eq. (19), and as before, decompose the constant matrix $(\tilde{A}-KC)$ into a diagonal and a strictly upper right triangular matrix D and N, respectively,

$$(\tilde{A} - KC) = U(D+N)U^{\dagger}.$$

The eigenvalues contained in D are determined by the choice of K, the form of the matrix N and the size of $|N|$ can be influenced by choosing A_{13} and A_{32} appropriately. For most choices of those two elements, the trajectories in phase space will be attracted to a limit cycle and these are the only choices we will consider. That system (20) has a limit cycle attractor is not surprising because for $p=1$ and under proper coordinate transformation, it can be written alternatively as

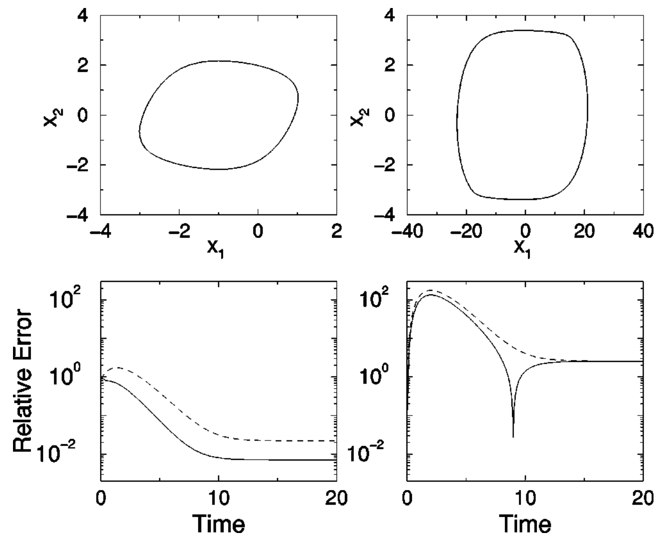

FIG. 2. The projection of the attractor of Eq. (20) onto the (x_1,x_2) plane and the relative error together with the error bound (13) are shown for the case $|N|\approx 2$ on the left and $|N|\approx 100$ on the right.

$$\ddot{\nu} + (\nu^2 - 1)\dot{\nu} + \nu = A_{13}\xi,$$

$$\dot{\xi} + \xi = -A_{32}\nu.$$

It is simply an augmented van der Pol oscillator. However, in the form (20), it can be seen to be a Lur'e system so that the synchronization may be achieved by designing a second system of the form (16). For simplicity, we choose p to be the only mismatched parameter, which means that the error dynamics are of the form (19) with the driving term h being time independent and given by the mismatch $\Delta p = 0.01$

We chose K in all examples in a way that the eigenvalues of $(\tilde{A}-KC)$ are $\lambda_i = -1$, with $i=1,2,3$. The size of the oscillations change with the choice of A_{13} and A_{32}, as can be seen from the projection of the limit cycle onto the (x_1,x_2) plane in Fig. 2, where on the left we chose $|N|\approx 2$ and on the right $|N|\approx 100$. To adjust for the different dynamics of the driving system when changing the parameters, we do not report the absolute error but the relative one, where we normalize the error with respect to the maximum amplitude of the driver

$$e_{rel}(t) = \frac{|\tilde{x}(t) - x(t)|}{\max_t |x(t)|}. \tag{21}$$

Figure 2 displays the time development of the relative error for small and large off-diagonal elements on the left and right, respectively. The initial polynomial growth and subsequent exponential decay of the maximum of the error to its

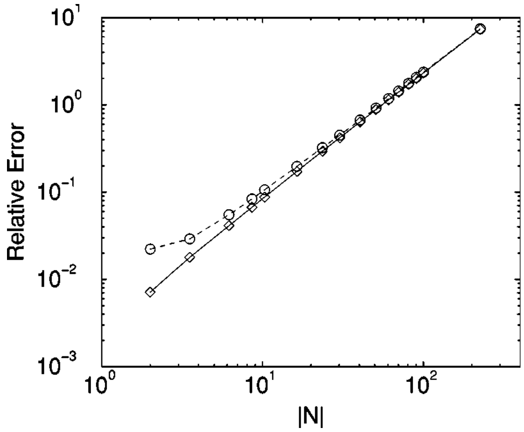

FIG. 3. This figure illustrates the dependence of the synchronization error on large off-diagonal elements in the matrix governing the error dynamics. We display the asymptotic relative error (solid line) and its bound (14) (dashed line) versus $|N|$.

asymptotic value as well as the considerable difference of the size of the relative error for the two cases can clearly be seen.

In Fig. 3 we vary the size of $|N|$ from about 2 to more than 200 while keeping the parameter mismatch, and therefore, the maximum of the driving term fixed at $h_{max}=0.01$. The asymptotic value of the relative error and the bound are displayed.

The asymptotic error $e_\infty=\lim_{t\to\infty}e(t)$ is in this case time independent because we chose the parameter mismatch so as to make h time independent. Let us denote this asymptotic error by e_∞. Then in this particular simple case the mismatch of the parameter leads to a shift of the synchronization manifold away from the identical synchronization manifold given in the full phase space with coordinates $(x,\tilde{x})$ by $\tilde{x}-x=0$ to $\tilde{x}-x-e_\infty=0$. For time dependent driving $h(x(t))$ (and under the assumption of normal hyperbolicity) the identical synchronization manifold will not be just shifted but distorted as well.

In many cases of practical interest the matrix A will not have large off-diagonal elements and the error will be small (of the same order as H). However, if one encounters a case where $|N|$ is large, then one has to take utmost care to match the parameters in the driver and the receiver because any small but finite mismatch can cause large synchronization errors, which might be even larger than the oscillations of the driver dynamics, as can be seen from Eq. (19) and Fig. 3.

IV. AN EXAMPLE: THE LORENZ SYSTEM

In the preceding section we restricted ourselves to Lur'e systems and the standard way of coupling them. The error bound was derived from Eq. (8) and we found a significant dependence of the synchronization error on the singular values of the linearized part of the error dynamics. We will show that this phenomenon is not restricted to the simple systems for which we presented an analytic bound, but can also be observed in chaotic systems where the error dynamics is given by Eq. (6).

Before we present an example of Eq. (6) based on the Lorenz system let us mention that two stability questions can be addressed in connection with Eq. (6). The first one reflects the *internal dynamics* of the nonlinear system and is known as Lyapunov stability. This means asymptotical stability of the equilibrium state $e=0$ in the case of no parameter mismatch between drive and response [that is, when $h(t)\equiv 0$]. The second question reflects the *external dynamics* of Eq. (6) and is known as bounded input bounded output (BIBO) stability. The system (6) is BIBO stable if any bounded input h produces a bounded output e. In general, Lyapunov stability does not guarantee BIBO stability, and vice versa. Desoer and Liu [14] have presented an example in which the state $e=0$ of the unforced system (6), with $h(t)\equiv 0$, is asymptotically stable in large, but nevertheless for bounded input (6) has an unbounded zero-state response, that is, its solution based on the initial condition $e(0)=0$ approaches infinity, when $t\to\infty$. In the cases considered in this paper, the error dynamics are Lyapunov stable and the dynamics of both the drive and the response system coupled to the drive are bounded, which implies BIBO stability for the error dynamics. Moreover, we assume that the mismatch of the parameters in Eq. (1) is small (but not arbitrarily small), and therefore $|h(t)|_{max}\leq H$ is also small.

We now present examples for which the synchronization manifold of a chaotic system is stable but nevertheless the synchronization error is large. For our examples we use a Lorenz system,

$$\begin{aligned}\dot{x}_1&=\sigma(-x_1+x_2),\\ \dot{x}_2&=\rho x_1-x_2-x_1x_3,\\ \dot{x}_3&=-bx_3+x_1x_2,\end{aligned}\tag{22}$$

which drives another Lorenz system with different parameters through diffusive coupling $K\cdot(\tilde{x}-x)$. If we write $\delta_1=\tilde{\sigma}-\sigma$, $\delta_2=\tilde{\rho}-\rho$, and $\delta_3=\tilde{b}-b$, then the equation for the error dynamics has the form of Eq. (6),

$$\dot{e}=[\tilde{A}(t)-K]e+h(t)+g(t,e),\tag{23}$$

with

$$\tilde{A}(t)=\begin{pmatrix}-\tilde{\sigma} & \tilde{\sigma} & 0\\ \tilde{\rho}-x_3 & -1 & -x_1\\ x_2 & x_1 & -\tilde{b}\end{pmatrix},\quad K=\begin{pmatrix}0 & \tilde{\sigma} & 0\\ k & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

and

$$h(t)=\begin{pmatrix}\delta_1[-x_1+x_2]\\ \delta_2 x_1\\ -\delta_3 x_3\end{pmatrix},\quad g(t,e)=\begin{pmatrix}0\\ -e_1e_3\\ e_1e_2\end{pmatrix}.$$

The reason for choosing the matrix K as shown is that this choice allows us to prove the global stability of the synchro-

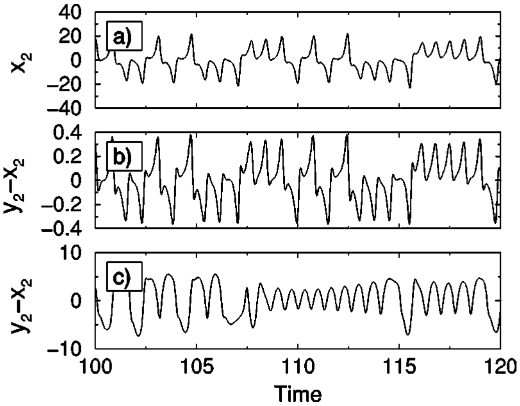

FIG. 4. (a) x_2 variable of the driving Lorenz system versus time; (b) difference y_2-x_2 between the transmitter and the receiver system for $k=0$ and a 1% mismatch of the parameters σ, ρ, and b; and (c) same as (b) with $k=400$ (note the different scale of the vertical axes).

nization manifold $e=0$ for the unforced system (23) for all driving trajectories (trajectories of the transmitter) [15]. Indeed, let $h(t)=0$ (i.e., we consider identical synchronization). Note first that since $\dot{e}_1=-\sigma e_1$, e_1 converges to zero. Therefore, the remaining two-dimensional system describing the evolution of e_2 and e_3 in the limit $t\to\infty$ can be written as

$$\dot{e}_2=-e_2-e_3x_1(t),$$

$$\dot{e}_3=-be_3+x_1(t)e_2.$$

Using the Lyapunov function $L=e_2^2+e_3^2$, one can show that $\dot{L}=-2(e_2^2+be_3^2)<0$. This means that the zero solution is asymptotically stable. Thus all normal (conditional) Lyapunov exponents are negative for all driving trajectories and condition (3) is satisfied.

We choose the values $\sigma=10$, $b=8/3$, and $\rho=28$ for the driver and parameters of the response, which differ by 1%. Figure 4 shows our results for two values of the parameter k, $k=0$, and $k=400$. One can see that when $k=400$, the synchronization error for the second variable of the Lorenz system is large, although the normal Lyapunov exponents for all driving trajectories are negative with a magnitude of more than one.

For this example, an upper bound of the driving term is given by $H=5.7$, the largest normal Lyapunov exponent is due to the structure of $A(t)=\tilde{A}(t)-K$, largely independent of k and given by $\lambda_{\perp}^{max}\approx-1.8$. The maximum synchronization error $|e(t)|_{max}$, however, increases with k as it is shown in Fig. 5, where we as well display $H/-\lambda_{max}$ for comparison. The error obtained by integration of the linearized Eq. (23) [where the $g(t,e)$ term is dropped] is essentially indistinguishable from the error we get by integrating the full equation. This shows that linearization is justified for the whole range of k and that the observed increase of $|e(t)|_{max}$ with k is caused by the changing properties of $A(t)$. Clearly, a proper bound for the case of time varying A should as well have some prefactor that takes the effect of the deformation

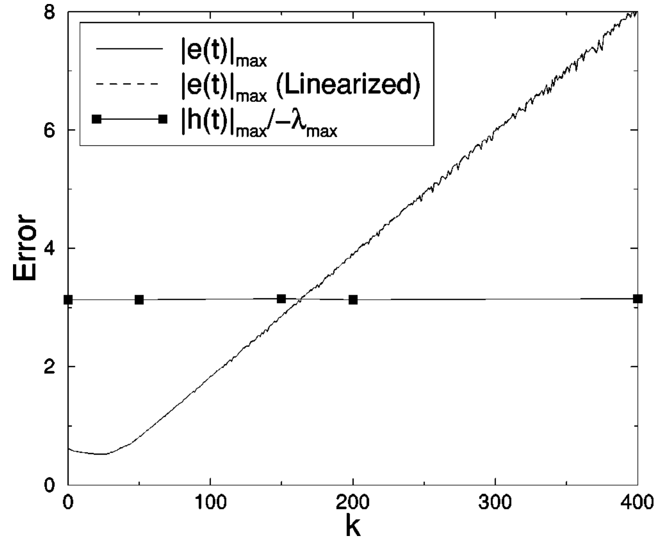

FIG. 5. Synchronization error $|e(t)|_{max}$ versus the coupling parameter k. We display $(H/-\lambda_{max})$ and the error obtained by integrating the linearized equation (23) for comparison. Linearization hardly changes the result, so that the dashed line nearly perfectly coincides with the solid line (error is obtained by integrating the full equation).

of the IS manifold into account. From the example we see that it scales with the size of the large off-diagonal element, and hence the largest singular value similar to what we observed in the case of a constant matrix A.

Note, in this and the next example that the condition guaranteeing normal hyperbolicity (4) is not met so that the GS manifold might be fractal. One can enforce Eq. (4) by having large diagonal elements in the coupling matrix. We checked and found that this did not change the results and therefore omitted it for simplicity.

The large synchronization error in the above example is due to the way we coupled the two Lorenz systems. We therefore present here a slightly modified example, where the synchronization error is large due to the size of an internal parameter of the transmitter. Let k be zero and increase the parameters ρ and $\tilde{\rho}$ of the two Lorenz systems starting from the nominal value $\rho=28$ and keeping the mismatch at 1%. Let us furthermore define the average synchronization error as

$$E\equiv\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T}\|e(s)\|ds, \tag{24}$$

where t_0 is chosen large enough to avoid all transient behavior and T is as large as feasible in terms of integration time. The quality of the synchronization depends in this case not on the value of the coupling constant but on the internal parameter ρ. We want to measure the error due to the deformation of the identical synchronization manifold as the two Lorenz systems go from the chaotic regime to one where stable limit cycles exist. Since in this case the amplitude of the oscillations in the driver changes with the parameter ρ, we normalize the synchronization error E by the error E_0,

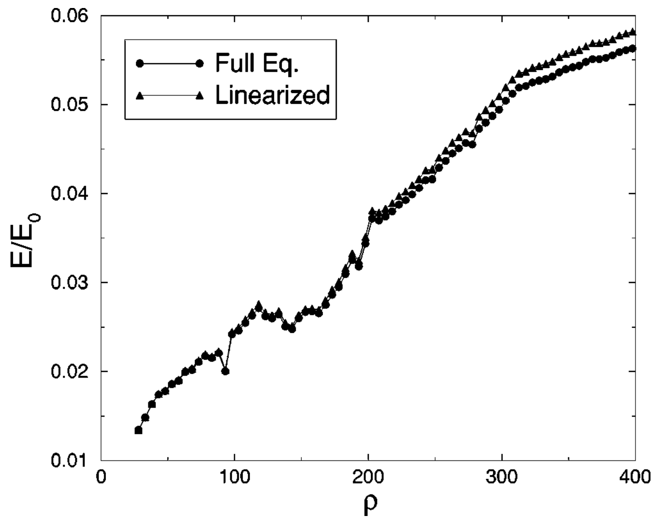

FIG. 6. Normalized average synchronization error versus internal parameter ρ for given relative parameter mismatch of 1%. We also display the error obtained by integrating the linearized equation (23).

which is the synchronization error of the drive and the response system being completely uncoupled. The normalized average synchronization error clearly grows with the size of the off-diagonal element as can be seen in Fig. 6.

V. CONCLUSION

In this paper we have shown both numerically and analytically criteria for stability and/or smoothness of synchronization manifolds (in terms of normal hyperbolicity) are not sufficient to guarantee a small synchronization error. For a simple subclass of systems (including the Lur'e systems), an analytic bound is presented. Numerical and analytical estimates of the synchronization error show that the bound for the deformation of the synchronization manifold grows when the largest singular value of the linearized error dynamics increases. Such large singular values occur (for fixed and stable eigenvalues) in cases with almost parallel eigenvectors, i.e., with non-normal matrices governing the dynamics of the synchronization error.

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APPENDIX A

We want to give here the derivation of the bound on $|e^{At}|$ needed to understand the behavior of $|x(t)|$. We suppose A is a stable square matrix, meaning that all the eigenvalues of A have negative real parts. Applying the Schur decomposition $U^{\dagger}AU=D+N$, we obtain

$$|e^{At}|=|e^{(D+N)t}|.$$

Since D and N do not commute, in general $e^{(D+N)t}\neq e^{Dt}e^{Nt}$. Setting $X(t):=e^{(D+N)t}$, we obtain the differential equation

$$\dot{X}(t)=DX(t)+NX(t),\quad X(0)=I,$$

which yields the following representation as an integral equation:

$$X(t)=e^{Dt}+\int_0^t e^{D(t-s)}NX(s)ds.$$

We solve this equation by Picard iteration, i.e., by a sequence of functions given by the iterative equation

$$X_{n+1}(t)=e^{Dt}+\int_0^t e^{D(t-s)}NX_n(s)ds,\quad X_0(t)=e^{Dt},$$

and use the ansatz $X_n(t)=e^{Dt}\Sigma_{k=0}^{n}\Theta_k(t)$ to get

$$e^{Dt}\Theta_{n+1}(t)=\int_0^t e^{D(t-s)}Ne^{Ds}\Theta_n(s)ds,\quad \Theta_0(t)=I.$$

It is easy to see that only $\Theta_0(t),\ldots,\Theta_{d-1}(t)$ are nonzero, since $e^{D(t-s)}$ is diagonal and N is strictly upper right triangular. This yields $X(t)=e^{Dt}\Sigma_{k=0}^{d-1}\Theta_k(t)$ and the estimate $|X(t)|\leqslant\Sigma_{k=0}^{d}|e^{Dt}\Theta_k(t)|$. Setting $\epsilon_k(t):=|e^{Dt}\Theta_k(t)|$, we have

$$\epsilon_{k+1}(t)\leqslant\int_0^t e^{\eta(t-s)}|N|\epsilon_k(s)ds,$$

from which we inductively obtain $\epsilon_k(t)\leqslant e^{\eta t}(|N|^k/k!)t^k$ and finally estimate

$$|X(t)|=|e^{At}|\leqslant e^{\eta t}\sum_{k=0}^{d-1}\frac{|N|^k}{k!}t^k.$$

More stringent estimates can be derived, but those result in a loss of simplicity of the expression and are therefore less useful for gaining insight into the reasons for the appearance of large errors.

APPENDIX B

Finally we give an estimate for $|N|$. Using Shur decomposition we obtain $U^{\dagger}AU=D+N$. This gives $|A|\leqslant|D|$

$|A| \leqslant |D| + |N|$ or $|N| \geqslant |A| - |D|$. On the other hand, we have $U^{\dagger}AU - D = N$, yielding $|A| + |D| \geqslant |N|$. We remark that $|A|$ is equal to the largest singular value $\sigma_{\max}$ of A. Setting $\rho := |D|$, which is equal to the eigenvalue of D with largest modulus, we have

$$\sigma_{\max} + \rho \geqslant |N| \geqslant \sigma_{\max} - \rho.$$

If the largest singular value $\sigma_{\max}$ of A is large compared to ρ, we see that $|N| \cong \sigma_{\max}$.

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